

A non-existence result on symplectic semifield spreads

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Abstract

We prove that there do not exist non-Desarguesian symplectic semifield spreads of $\text{PG}(5, q^2)$, $q \geq 2^{14}$ even, whose associated semifield has center containing \mathbb{F}_q , by proving that the only \mathbb{F}_q -linear set of rank 6 disjoint from the secant variety of the quadric Veronese variety of $\text{PG}(5, q^2)$ is a plane with three points of the Veronese surface of $\text{PG}(5, q^6) \setminus \text{PG}(5, q^2)$.

1 Introduction

Let $\text{PG}(n-1, q)$ be the projective space of dimension $n-1$ over the finite field \mathbb{F}_q of order q . Let $M(n, q)$ be the set of $n \times n$ matrices over \mathbb{F}_q .

A *planar spread* \mathcal{S} of $\text{PG}(2n-1, q)$, which we will call simply *spread* from now on, is a partition of the point-set in $(n-1)$ -dimensional subspaces. With any spread \mathcal{S} it is associated a translation plane $A(\mathcal{S})$ of order q^n in the following way: embed $\text{PG}(2n-1, q)$ in $\text{PG}(2n, q)$ as a hyperplane section, then the points of $A(\mathcal{S})$ are the points of $\text{PG}(2n, q) \setminus \text{PG}(2n-1, q)$, the lines are the n -dimensional subspaces of $\text{PG}(2n, q)$ intersecting $\text{PG}(2n-1, q)$ in an element of \mathcal{S} and the incidence is containment (see e.g. [4, Section 5.1]). Translation planes associated with different spreads are isomorphic if and only if there is a collineation of $\text{PG}(2n-1, q)$ mapping one spread to the other (see [1] or [15, Chapter 1]). Without loss of generality, we may always assume that $S(\infty) := \{(\mathbf{0}, \mathbf{y}), \mathbf{y} \in \mathbb{F}_q^n\}$ and $S(0) := \{(\mathbf{x}, \mathbf{0}), \mathbf{x} \in \mathbb{F}_q^n\}$ belong to \mathcal{S} , hence we may write $\mathcal{S} = \{S(A), A \in \mathbb{C}\} \cup S(\infty)$, with $S(A) :=$

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$\{(\mathbf{x}, \mathbf{x}A), \mathbf{x} \in \mathbb{F}_q^n\}$ and $\mathbb{C} \subset M(n, \mathbb{F}_q)$ such that $|\mathbb{C}| = q^n$, \mathbb{C} contains the zero matrix and $A - B$ is non-singular for every $A, B \in \mathbb{C}$. The set \mathbb{C} is called the spread set associated with \mathcal{S} .

A spread \mathcal{S} is said to be *Desarguesian* if $A(\mathcal{S})$ is isomorphic to $\text{AG}(2, q^n)$ and hence a plane coordinatized by the field of order q^n . The spread \mathcal{S} is said to be a *semifield spread* if $A(\mathcal{S})$ is a plane of Lenz-Barlotti class V and this is equivalent to saying that $A(\mathcal{S})$ is coordinatized by a semifield. A finite *semifield* $(F, +, *)$ is a finite nonassociative division algebra. If $(F, +, *)$ satisfies all the axioms for a semifield except, possibly, the existence of an identity element for the multiplication, then it is called a *presemifield*. A semifield spread \mathcal{S} is such that there exists an elementary abelian subgroup G of $\text{PGL}(2n, q)$ of order q^n fixing an element $X \in \mathcal{S}$ point-wise and acting regularly on $\mathcal{S} \setminus X$. If we set $X = S(\infty)$, then \mathbb{C} turns out to be closed under addition; hence \mathbb{C} is a set of q^n $n \times n$ invertible matrices over \mathbb{F}_q that form a subgroup of the additive group of $M(n, \mathbb{F}_q)$ ([4, Section 5.1]). Then \mathbb{C} is a vector space over some subfield of \mathbb{F}_q . This has led to the following geometric interpretation (see [13] as it first appeared for $n = 2$ and [11] for the general case). Let $\text{PG}(n^2 - 1, q)$ be the projective space induced by $M(n, \mathbb{F}_q)$ and let \mathcal{D} be the algebraic variety of $\text{PG}(n^2 - 1, q)$ consisting of the singular matrices; \mathcal{D} is a so-called determinantal variety and it is the $(n - 2)$ -th secant variety of the Segre variety $\Sigma_{n-1, n-1} := \text{PG}(n - 1, q) \times \text{PG}(n - 1, q)$ (see [5, Ch.9]). If \mathbb{C} is an \mathbb{F}_s -vector space, $q = s^t$, then $\dim_{\mathbb{F}_s} \mathbb{C} = nt$ and it defines a subset Λ of $\text{PG}(n^2 - 1, q)$ called \mathbb{F}_s -linear set of rank nt (for a beautiful and complete overview of the topic see [17]). So finding a semifield spread of $\text{PG}(2n - 1, q)$ (and hence a semifield plane of order q^n) is equivalent to finding an \mathbb{F}_s -linear set of $\text{PG}(n^2 - 1, q)$, $q = s^t$, of rank nt disjoint from \mathcal{D} . We recall that the left nucleus N_l of a semifield F is the set $\{k \in F | k * (x * y) = (k * x) * y, \forall x, y \in F\}$ and the center is the set $\mathcal{K} = \{k \in F | k * (x * y) = (k * x) * y, x * (k * y) = (k * x) * y, x * (y * k) = (x * y) * k \quad \forall x, y \in F\}$, so an \mathbb{F}_s -linear set of $\text{PG}(n^2 - 1, q)$ of rank nt disjoint from \mathcal{D} , with \mathbb{F}_s maximum subfield of linearity leads to a semifield of order q^n , center isomorphic to \mathbb{F}_s and left nucleus isomorphic to \mathbb{F}_q . In [11], the following more general result has been proved: two semifields of order q^n , with left nucleus containing \mathbb{F}_q and center containing \mathbb{F}_s , are isotopic (that is they coordinatize isomorphic translation planes) if and only if there is collineation fixing the two systems of maximal subspaces of $\Sigma_{n-1, n-1} \subset \text{PG}(n^2 - 1, q)$ mapping one \mathbb{F}_s -linear set to the other.

The spread \mathcal{S} is said to be *symplectic* if the elements of \mathcal{S} are totally isotropic with respect a *symplectic polarity* of $\text{PG}(2n - 1, q)$. Let $\beta((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)) = \mathbf{x}_1 \mathbf{y}_2^T - \mathbf{y}_1 \mathbf{x}_2^T$, $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}_q^n$, then β is an alternating bilinear form of $\text{PG}(2n - 1, q)$ and it induces a symplectic polarity \perp . The subspace $S(\infty)$ is clearly totally isotropic with respect to \perp . The subspace $S(A) \in \mathcal{S}$ is totally isotropic

if and only if $\mathbf{x}_1(\mathbf{x}_2 A)^T - (\mathbf{x}_1 A)\mathbf{x}_2^T = 0 \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_q^n$ and this is true if and only if A is symmetric. Hence to a symplectic spread \mathcal{S} it is possible to associate a spread set \mathbb{C} consisting of symmetric matrices (see e.g. [9, 16]). The symmetric matrices form a subspace of $M(n, \mathbb{F}_q)$ of dimension $\frac{n(n+1)}{2}$ that then induces a subspace $\text{PG}(\frac{n(n+1)}{2} - 1, q)$ of $\text{PG}(n^2 - 1, q)$. The rank 1-symmetric matrices are the Veronese variety \mathcal{V} of degree 2 of $\text{PG}(\frac{n(n+1)}{2} - 1, q)$ (this is the so called determinantal representation of the Veronese variety of degree 2, see [5, Example 2.6]). Hence the singular symmetric matrices form the $(n - 2)$ -th secant variety, say \mathcal{V}_{n-2} of the Veronese variety. So to a symplectic semifield spread of $\text{PG}(2n - 1, q)$ there corresponds an \mathbb{F}_s -linear set Λ , $q = s^t$, of $\text{PG}(\frac{n(n+1)}{2} - 1, q)$ of rank tn such that $\Lambda \cap \mathcal{V}_{n-2} = \emptyset$ (see also [14]).

For odd q , there are many examples of non-Desarguesian symplectic semifield spreads of $\text{PG}(2n - 1, q)$, also due to the connection between DO polynomials and commutative semifields of odd order (by [9], any symplectic semifield defines a presemifield isotopic to a commutative semifield).

When q is even, a symplectic spread of $\text{PG}(2n - 1, q)$ gives rise to many interesting geometric structures. For even q , the symplectic polar space of $\text{PG}(2n - 1, q)$ is isomorphic to the orthogonal space of $\text{PG}(2n, q)$ induced by a non-singular parabolic quadric $\mathcal{Q}(2n, q)$ as, for even q , $\mathcal{Q}(2n, q)$ has a nucleus N through which all the tangent lines pass: projecting the points of $\mathcal{Q}(2n, q)$ from N to a hyperplane, we obtain an incidence structure that is a symplectic polar space. Hence a symplectic spread \mathcal{S} of $\text{PG}(2n - 1, q)$ is equivalent to a spread of $\mathcal{Q}(2n, q)$, i.e., a partition of the point-set of $\mathcal{Q}(2n, q)$ in subspaces of maximum dimension $n - 1$. If n is odd, a spread of $\mathcal{Q}(2n, q)$ in turn leads to a spread of a hyperbolic quadric $\mathcal{Q}^+(2n + 1, q)$: embed $\mathcal{Q}(2n, q)$ in $\mathcal{Q}^+(2n + 1, q)$ as a hyperplane section, then a maximal subspace of $\mathcal{Q}(2n, q)$ is contained in two maximal subspaces of $\mathcal{Q}^+(2n + 1, q)$, one for each system, so if we pick one, we obtain a spread of $\mathcal{Q}^+(2n + 1, q)$, i.e., a partition of the point-set of $\mathcal{Q}^+(2n + 1, q)$ in subspaces of dimension n . By a suitable choice of coordinates, every such a spread gives rise to a Kerdock set, i.e., a set of q^n skew-symmetric $(n + 1) \times (n + 1)$ matrices, n odd, such that the difference of any two of them is nonsingular. From such a Kerdock set it is possible to obtain a remarkable class of codes: namely, binary Kerdock codes of length 2^{n+1} (see [3]). By slicing a spread of $\mathcal{Q}^+(2n + 1, q)$ in nonequivalent hyperplanes under the action of the subgroup of the orthogonal group fixing it, we obtain nonequivalent spreads of $\mathcal{Q}(2n, q)$, called *cousins* in [7], and hence, for even q , nonequivalent symplectic spreads and translation planes. Hence starting from one symplectic spread it is possible to get many more (see [7]).

In the particular case when $n = 3$, we obtain a spread of $\mathcal{Q}^+(7, q)$ and hence, by triality, we obtain an ovoid of $\mathcal{Q}^+(7, q)$, a combinatoric structure that gives rise to many others (see [7, 8]).

In this article, we are focused on symplectic semifield spreads of $\text{PG}(5, q)$, when q is even. In such a case, only two nonsporadic examples are known: the Desarguesian spread and one of its cousin (see [7]), so they are both obtained by slicing the so called Desarguesian spread of $\mathcal{Q}^+(7, q)$. In the first case, the associated translation plane is the Desarguesian plane, hence it is coordinatized by the finite field of order q^3 and the relevant linear set is actually linear on \mathbb{F}_q . In the second case we have, somehow, the “opposite situation”: the semifield spread is associated to a spread set \mathbb{C} that gives rise to an \mathbb{F}_2 -linear set Λ of $\text{PG}(5, q)$, where \mathbb{F}_2 is the maximum subfield of \mathbb{F}_q for which Λ is linear, hence the associate semifield has order q^3 and center \mathbb{F}_2 . In this article we prove the following:

Main result. *The only \mathbb{F}_q -linear set Λ of rank 6 disjoint from the secant variety of the Veronese surface of $\text{PG}(5, q^2)$, for even $q \geq 2^{14}$, is a plane with three points of the Veronese surface of $\text{PG}(5, q^6) \setminus \text{PG}(5, q^2)$, and hence there do not exist non-Desarguesian semifield symplectic spreads of $\text{PG}(5, q^2)$, whose associated semifield has center containing \mathbb{F}_q .*

2 Quadric Veronesean and its secant variety

In this section we denote by \mathbb{P}^{n-1} the $(n - 1)$ -dimensional projective space over the generic field \mathbb{F} .

The Veronese map of degree 2

$$v_2 : (x_0, x_1, \dots, x_{n-1}) \in \mathbb{P}^n \longmapsto (\dots, \mathbf{x}^l, \dots) \in \mathbb{P}^{N-1}$$

is such that \mathbf{x}^l ranges over all monomials of degree 2 in x_0, x_1, \dots, x_{n-1} , hence $N = \frac{n(n+1)}{2}$. The image of v_2 is an algebraic variety called quadric Veronese variety. If we use the so-called determinantal representation of the Veronese variety of degree 2 (see [5, Example 2.6]), then \mathbb{P}^{N-1} is induced by the subspace of $M(n, \mathbb{F})$ consisting of symmetric matrices and, by $v_2(x_0, x_1, \dots, x_{n-1}) = A$ such that $a_{ij} = x_i x_j$, the Veronese variety turns out to be the intersection of $\Sigma_{n-1, n-1} = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ with such a \mathbb{P}^{N-1} . We recall that the Segre variety $\Sigma_{n-1, n-1} \subset \mathbb{P}^{n^2-1}$ consists of all rank 1 matrices of $M(n, \mathbb{F})$ and the k th secant variety of $\Sigma_{n-1, n-1}$, i.e., the union of k -subspaces spanned by points of $\Sigma_{n-1, n-1}$, is the algebraic variety consisting of the matrices of rank at most $k + 1$ (see [5, Example 9.2]), hence the k th secant variety of the Veronese variety consists of the symmetric matrices of rank at most $k + 1$.

We are interested in the particular case when $n = 3$ and hence the quadric Veronese surface is $\mathcal{V} = v_2(\mathbb{P}^2) = \left\{ \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}, x, y, z \in \mathbb{F} \right\} \subset \mathbb{P}^5$.

The image of a line ℓ of \mathbb{P}^2 is a conic, intersection of \mathcal{V} with a suitable plane. If the characteristic of \mathbb{F} is even, then a line of the plane is tangent to a conic if and only if it contains a fixed point, called the nucleus of the conic. Then, when the field characteristic is even, there exists a plane π_N of \mathbb{P}^5 , called the nucleus plane, such that the nucleus of each conic $v_2(\ell)$ belongs to π_N and vice versa each point of π_N is a nucleus of one and only one conic $v_2(\ell)$ (see [6, Th.25.1.17] for this particular case and [18] for the general case). In the representation we have chosen, we have

$$\pi_N = \left\{ \begin{pmatrix} 0 & t & u \\ t & 0 & v \\ u & v & 0 \end{pmatrix}, t, u, v \in \mathbb{F} \right\}.$$

The automorphism group \hat{G} of \mathcal{V} is the lifting of $G = PGL(3, \mathbb{F})$ acting in the obvious way: $v_2(P)^{\hat{g}} = v_2(P^g)$, $\forall g \in PGL(3, \mathbb{F})$.

The secant variety \mathcal{V}_1 of \mathcal{V} is the variety of \mathbb{P}^5 consisting of the points lying on a line secant to \mathcal{V} . It is a determinantal variety, i.e., it consists of

the symmetric 3×3 matrices $\left\{ \begin{pmatrix} t_0 & t_3 & t_4 \\ t_3 & t_1 & t_5 \\ t_4 & t_5 & t_2 \end{pmatrix}, t_i \in \mathbb{F}, i = 0, 1, \dots, 5 \right\}$ with

zero determinant. So \mathcal{V}_1 is a hypersurface of \mathbb{P}^5 with equation $t_0 t_1 t_2 - t_0 t_5^2 - t_1 t_4^2 - t_2 t_3^2 + 2 t_3 t_4 t_5 = 0$. The automorphism group \hat{G} of \mathcal{V} obviously fixes \mathcal{V}_1 . It is well known that the singular points of \mathcal{V}_1 are the points of \mathcal{V} (see, e.g., [5, Exercise 14.15]), that are actually double points. Moreover, it is easy to see that, when the field characteristic is even, every tangent hyperplane contains π_N and viceversa. We observe that $\pi_N \subset \mathcal{V}_1$.

Proposition 1. *Let \mathcal{V}_1 be the secant variety of the quadric Veronese variety $\mathcal{V} = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$. The maximal linear subspaces contained in \mathcal{V}_1 are planes and they are: the planes intersecting $v_2(\mathbb{P}^2)$ in $v_2(\ell)$, with ℓ a line of \mathbb{P}^2 , the planes tangent to the Veronese variety and, if the field characteristic is even, the nucleus plane.*

Proof. The maximal linear subspaces contained in the determinantal variety of the 3×3 singular matrices are of four types: the matrices with a common kernel, the matrices with a common image, the matrices A such that $A(V) \subset W$, with V and W subspaces of \mathbb{F}^3 of dimension 2 and 1 respectively (here by dimension we mean vector space dimension), and the skew-symmetric

matrices (with zero diagonal) (see [5, page 113]). The variety \mathcal{V}_1 is the variety of the symmetric 3×3 singular matrices and it is easy to see that we can get subspaces of projective dimension at most 2. Moreover, the matrices of the first two types coincide. Let L be the linear subspace of the symmetric singular matrices A such that $(a, b, c)A = \mathbf{0}$ for some fixed non-zero vector $(a, b, c) \in \mathbb{F}^3$. Consider the line $\ell := \{(x, y, z) \in \mathbb{P}^2 \mid ax + by + cz = 0\}$, then $v_2(\ell)$ is clearly contained in L . Let now L be linear subspace of the symmetric singular matrices A such that $A(V) \subset W$, with V and W subspaces of \mathbb{F}^3 of dimension 2 and 1 respectively. Let $w_1, w_2 \in \mathbb{F}^3$ two vectors defining W , i.e. $w_i \cdot w = 0 \forall w \in W$, and let v_1, v_2 be two independent vectors of V , then we have $L = \{A \text{ singular and symmetric} \mid v_i A(w_j)^T = 0 \forall i = 1, 2\}$. In order to have a linear space of dimension 3, we must have $\{v_1, v_2\} = \{w_1, w_2\}$, hence $L = \{A \text{ singular and symmetric} \mid v_i A(v_j)^T = 0, i = 1, 2\}$. Let $v_i = (a_i, b_i, c_i), i = 1, 2$ and let $A \in \mathcal{V}$, then $v_i A(v_j)^T = (a_i x + b_i y + c_i z)(a_j x + b_j y + c_j z)$. Hence $L \cap \mathcal{V}$ is the unique point $(x, y, z) \in \mathbb{P}^2$ such that $(a_1 x + b_1 y + c_1 z) = (a_2 x + b_2 y + c_2 z) = 0$ and it is the tangent plane to \mathcal{V} at $v_2(x, y, z)$. The skew-symmetric matrices (with zero diagonal) belong to \mathcal{V}_1 if and only if the field characteristic is even and they form the nucleus plane π_N . \square

In the following, we will refer to the planes containing $v_2(\ell)$, with ℓ a line of \mathbb{P}^2 , as *conic planes* and to the planes tangent to \mathcal{V} as *tangent planes*.

Proposition 2. [6, Theorem 25.1.17] *Let the field characteristic be even. Each conic plane intersects the nucleus plane in a point and each tangent plane intersects the nucleus plane in a line.*

Proposition 3. *Let \mathcal{V}_1 be the secant variety of the Veronese surface of $PG(5, q)$, $q \geq 2^8$, then a plane π is disjoint from \mathcal{V}_1 if and only if $\pi \cap \mathcal{V}_1$ consists of three lines through three points of the Veronese surface over \mathbb{F}_{q^3} .*

Proof. Over the algebraic closure of \mathbb{F}_q , $\pi \cap \mathcal{V}_1$ consists of a curve \mathcal{C} of degree 3. By [2, Corollary 7.4], if the curve of degree d is absolutely irreducible and $q > \max\{4 \cdot d^2, 2 \cdot d^4\}$, then it has at least one \mathbb{F}_q rational point. Hence, for $q \geq 2^8$, \mathcal{C} has no \mathbb{F}_q -rational points if and only if it is reducible and the only possibility is that \mathcal{C} consists of three non-concurrent lines over \mathbb{F}_{q^3} , say $\ell, \ell^\sigma, \ell^{\sigma^2}$, where σ is the \mathbb{F}_q -linear collineation of the plane induced by $Gal(\mathbb{F}_{q^3}/\mathbb{F}_q)$. Let P be $\ell \cap \ell^\sigma$, hence the points $P, P^\sigma, P^{\sigma^2}$ are singular for \mathcal{C} . Suppose that P is not a singular point for \mathcal{V}_1 , then π is contained in the tangent hyperplane of P . Since q is even, the tangent hyperplane contains also π_N , hence π and π_N intersects in at least one point and since the two planes are defined over \mathbb{F}_q , the point is \mathbb{F}_q -rational. As $\pi_N \subset \mathcal{V}_1$, we get a an

\mathbb{F}_q -rational point of $\pi \cap \mathcal{V}_1$. Hence $P, P^\sigma, P^{\sigma^2}$ have to be singular for \mathcal{V}_1 , i.e. P, P^σ and $P^{\sigma^2} \in \mathcal{V}$ and they are $\mathbb{F}_{q^3} \setminus \mathbb{F}_q$ -rational. \square

3 Proof of the main result

An \mathbb{F}_q -linear set Λ of rank m of $\text{PG}(n-1, q^2)$ is a set of points of $\text{PG}(n-1, q^2)$ defined by an m -dimensional vector space over \mathbb{F}_q . Let $P = (x_0, x_1, \dots, x_{n-1}) \in \Lambda$, then, by definition, $\lambda(x_0, x_1, \dots, x_{n-1}) \in \Lambda \forall \lambda \in \mathbb{F}_q$. If $\lambda(x_0, x_1, \dots, x_{n-1}) \in \Lambda \forall \lambda \in \mathbb{F}_{q^2}$, then we say that P has *weight 2*, otherwise P is said to be of *weight 1*. If each point of Λ has weight 1, then $\Lambda \cong \text{PG}(m-1, q)$. If Λ contains points of weight 2 and W is an m -dimensional vector space defining Λ , then we have $W = W_1 \oplus W_2$, with W_2 the maximal subspace of W that is a vector space also over \mathbb{F}_{q^2} , $\dim_{\mathbb{F}_q} W_1 = h$, $\dim_{\mathbb{F}_q} W_2 = 2k$, $h + 2k = m$, $\langle \Lambda \rangle$ is a $\text{PG}(h+k-1, q^2)$ and Λ is a cone with vertex a $\text{PG}(k-1, q^2)$ and base a $\text{PG}(h-1, q)$.

Throughout this section, we let q be even. If $f_i \in \mathbb{F}[x_0, x_1, \dots, x_{n-1}]$, $i \in \mathcal{I}$, then let $V(f_i, i \in \mathcal{I}) \subset \mathbb{P}^{n-1}$ be the algebraic variety consisting of the solutions of $f_i = 0 \forall i \in \mathcal{I}$.

Theorem 4. *There does not exist an \mathbb{F}_q -linear set Λ of rank 6 disjoint from \mathcal{V}_1 in $\text{PG}(5, q^2)$ for $q \geq 2^{14}$, unless Λ is a plane of $\text{PG}(5, q^2)$.*

Proof. The variety \mathcal{V}_1 is a hypersurface of degree 3 consisting of the zeros of $t_0 t_1 t_2 + t_0 t_5^2 + t_1 t_4^2 + t_2 t_3^2$. The nucleus plane $\pi_N := \{(0, 0, 0, t_3, t_4, t_5), t_i \in \mathbb{F}_{q^2}, i = 3, 4, 5\}$ is contained in \mathcal{V}_1 so we must have $\Lambda \cap \pi_N = \emptyset$ in $\text{PG}(5, q^2)$ and hence $\Lambda = \{(x, y, z, F_1, F_2, F_3), x, y, z \in \mathbb{F}_{q^2}\}$, with F_i \mathbb{F}_q -linear function of x, y, z . For $a \in \mathbb{F}_q$, let $t^2 + at + 1$ be an irreducible polynomial over \mathbb{F}_q and let ξ be one of its roots in \mathbb{F}_{q^2} . Then $\{1, \xi\}$ is a basis of \mathbb{F}_{q^2} as \mathbb{F}_q -vector space. Hence we can write $x = x_1 + x_2 \xi$, $y = y_1 + y_2 \xi$, $z = z_1 + z_2 \xi$, with $x_i, y_i, z_i \in \mathbb{F}_q, i = 1, 2$ and $F_i : \mathbb{F}_{q^2}^3 \rightarrow \mathbb{F}_{q^2}$ as $F_1(x_1, x_2, y_1, y_2, z_1, z_2) = l_1 + \xi l_2$, $F_2(x_1, x_2, y_1, y_2, z_1, z_2) = m_1 + \xi m_2$, $F_3(x_1, x_2, y_1, y_2, z_1, z_2) = n_1 + \xi n_2$, with l_i, m_i, n_i linear functions of $x_1, x_2, y_1, y_2, z_1, z_2$. In order to avoid confusion, we will denote by Σ the $\text{PG}(5, q^2)$ where we have defined \mathcal{V}_1 and by $\Sigma(\mathbb{F}_{q^{2n}})$ the geometry obtained by the field extension of degree n of \mathbb{F}_{q^2} , whereas we will denote by Π the $\text{PG}(5, q) = \{(x_1, x_2, y_1, y_2, z_1, z_2, l_1, l_2, m_1, m_2, n_1, n_2), x_i, y_i, z_i \in \mathbb{F}_q\}$ and by $\Pi(\mathbb{F}_{q^n})$ the projective space containing Π as subgeometry obtained by the field extension of degree n of \mathbb{F}_q . We remark that Π induces the linear set Λ in Σ .

A point in $\Lambda \cap \mathcal{V}_1$ has to fulfill:

$$(x_1 + x_2 \xi)(y_1 + y_2 \xi)(z_1 + z_2 \xi) + (x_1 + x_2 \xi)(n_1 + \xi n_2)^2 + (y_1 + y_2 \xi)(m_1 + \xi m_2)^2 + (z_1 + z_2 \xi)(l_1 + \xi l_2)^2 = (x_1 + x_2 \xi)(y_1 + y_2 \xi)(z_1 + z_2 \xi) + (x_1 + x_2 \xi)(n_1^2 + \xi^2 n_2^2) +$$

$$(y_1 + y_2\xi)(m_1^2 + \xi^2 m_2^2) + (z_1 + z_2\xi)(l_1^2 + \xi^2 l_2^2) = x_1 y_1 z_1 + \xi(x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1) + \xi^2(x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1) + \xi^3 x_2 y_2 z_2 + x_1 n_1^2 + y_1 m_1^2 + z_1 l_1^2 + i(x_2 n_1^2 + y_2 m_1^2 + z_2 l_1^2) + \xi^2(x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2) + \xi^3(x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2) = 0.$$

As ξ is a root of $t^2 + at + 1$, we have $\xi^2 = a\xi + 1$ and $\xi^3 = a\xi^2 + \xi = a(a\xi + 1) + \xi = (a^2 + 1)\xi + a$, so:

$$x_1 y_1 z_1 + \xi(x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1) + \xi^2(x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1) + \xi^3 x_2 y_2 z_2 + x_1 n_1^2 + y_1 m_1^2 + z_1 l_1^2 + \xi(x_2 n_1^2 + y_2 m_1^2 + z_2 l_1^2) + \xi^2(x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2) + \xi^3(x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2) =$$

$$x_1 y_1 z_1 + x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1 + a x_2 y_2 z_2 + x_1 n_1^2 + y_1 m_1^2 + z_1 l_1^2 + x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2 + a(x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2) + \xi(x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1 + x_2 n_1^2 + y_2 m_1^2 + z_2 l_1^2 + a(x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1 + x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2) + (a^2 + 1)(x_2 y_2 z_2 + x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2)) = 0,$$

implying:

$$\begin{cases} f := x_1 y_1 z_1 + x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1 + a x_2 y_2 z_2 + x_1 n_1^2 + y_1 m_1^2 + z_1 l_1^2 \\ + x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2 + a(x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2) = 0 \\ g := x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1 + x_2 n_1^2 + y_2 m_1^2 + z_2 l_1^2 + a(x_1 y_2 z_2 + x_2 y_1 z_2 \\ + x_2 y_2 z_1 + x_1 n_2^2 + y_1 m_2^2 + z_1 l_2^2) + (a^2 + 1)(x_2 y_2 z_2 + x_2 n_2^2 + y_2 m_2^2 + z_2 l_2^2) = 0 \end{cases}.$$

and the only solution for this system of equations must be $x_i, y_i, z_i = 0$

$\forall i = 1, 2$.

That is equivalent to asking that the algebraic variety $V(f, g)$ defined in $\text{PG}(5, q)$ (i.e., f and g polynomials with coefficients in \mathbb{F}_q) should not contain \mathbb{F}_q -rational points. Classical results, as the Lang-Weil bound ([10]), state that when V is an absolutely irreducible variety of dimension r defined by polynomials over the finite field \mathbb{F}_q , for q "big enough", $|V|$ is roughly q^r . More precisely, by [2, Corollary 7.4], if d is the degree of V , then for $q > \max\{2(r+1)d^2, 2d^4\}$ V has at least one \mathbb{F}_q -rational point. In our case $r \leq 4$ and $d \leq 9$, so for $q > 2 \cdot 9^4$, if $V(f, g)$ is absolutely irreducible, it has at least one point. Hence, assuming that $q \geq 2^{14}$, in order to get a variety with no \mathbb{F}_q -rational points, we must have a variety V which is reducible over some extension of \mathbb{F}_q , say \mathbb{F}_{q^t} , and such that $V = W \cup W^\sigma \cup \dots \cup W^{\sigma^{t-1}}$, W a (possible reducible) variety with the same dimension as V , $\langle \sigma \rangle = \text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and $W \cap W^\sigma \cap \dots \cap W^{\sigma^{t-1}} = \emptyset$, where, by abuse of notation, we denote by σ^i also the map $(x_0, x_1, \dots, x_{n-1}) \in \text{PG}(n-1, q^t) \mapsto (x_0^{\sigma^i}, x_1^{\sigma^i}, \dots, x_{n-1}^{\sigma^i}) \in \text{PG}(n-1, q^t)$, hence $P = (x_0, x_1, \dots, x_{n-1}) \in \text{PG}(n-1, q)$ if and only if $P^\sigma = P$. We remark that σ^i induces an automorphism on the projective space, hence W and W^{σ^i} have the same degree and dimension.

Let $V := V(f, g)$. Suppose that one of the two polynomials, say f , is reducible over \mathbb{F}_q , hence f has a linear factor f_1 over \mathbb{F}_q and $V(f_1, g) \subseteq V(f, g)$. As V does not contain \mathbb{F}_q -rational points, f_1 cannot be a factor of g and hence the variety $V(f_1, g)$ is a cubic hypersurface of $\text{PG}(4, q)$. By the Chevalley-Warning theorem, $V(f_1, g)$ has at least one point over \mathbb{F}_q . Hence both f and g are irreducible over \mathbb{F}_q . In the algebraic closure, $\dim V = 3$ unless both $V(f)$ and $V(g)$ are reducible varieties over some field extension and they have some common component. The hypersurface $V(f)$ is reducible if and only if f is reducible on some field extension and since all the components have the same degree, the only possibility is that $f = f_1 f_2 f_3$, where f_i is linear and defined over \mathbb{F}_{q^3} . The same is true for $V(g)$, hence $g = g_1 g_2 g_3$ and, in order to get $\dim V = 4$, we must have $f_i = g_j$ for some i and j , but then $f = g$ and this is obviously not the case.

So we may assume that $\dim V = 3$ and that $V = W \cup W^\sigma \cup \dots \cup W^{\sigma^{t-1}}$. We have that $\dim W = 3$ and let $\deg W$ be d and the multiplicity be μ (if W is reducible, then d and μ are the sum of the degrees and multiplicities of the irreducible components of W of dimension 3), so $d\mu t = \deg(f) \cdot \deg(g) = 9$ (see [5, Theorem 18.4]). If $t = 9$, then $d = 1$ and W is a 3-dimensional subspace over \mathbb{F}_{q^9} . Hence W induces a \mathbb{F}_{q^9} -linear set of rank 4 of $\Sigma(\mathbb{F}_{q^{18}})$. Such a linear set can be a $\text{PG}(3, q^9)$, a cone with vertex a point and base a subline $\text{PG}(1, q^9)$ (hence contained in a plane of $\Sigma(\mathbb{F}_{q^{18}})$), or a line of $\Sigma(\mathbb{F}_{q^{18}})$. In the first case, we would get a $\text{PG}(3, q^9)$ contained in \mathcal{V}_1 in $\Sigma(\mathbb{F}_{q^{18}})$, but this implies that there is a $\text{PG}(3, q^{18})$ contained in \mathcal{V}_1 , a contradiction to Proposition 1. Suppose that W induces a cone. If a plane $\text{PG}(2, q^{18})$ shares with \mathcal{V}_1 $q^9 + 1$ lines, then it is contained in \mathcal{V}_1 . Since the W^{σ^i} pairwise intersect in at least a line, the linear sets induced by them intersect in a \mathbb{F}_{q^9} -linear set of rank 2. A \mathbb{F}_{q^9} -linear set of rank 2 of $\text{PG}(2, q^{18})$ is either a point or a subline $\text{PG}(1, q^9)$. As the W^{σ^i} are contained in planes that are all conic or all tangent and the planes of the same type pairwise intersect in exactly a point (see, e.g., [6, Theorems 25.1.11 and 25.1.16]), the linear sets induced by the W^{σ^i} pairwise intersect in a point of weight 2. As in a cone there is only one point of weight 2, they all intersect in the same point, contradicting $W \cap W^\sigma \cap \dots \cap W^{\sigma^{t-1}} = \emptyset$. So assume that W induces a line. Again, these lines pairwise intersect in a point, but they are not through the same one, hence we get 9 lines in a plane. If a plane shares with \mathcal{V}_1 9 lines, then it is contained in \mathcal{V}_1 . Let π be such a plane, then $\Pi(\mathbb{F}_{q^9})$ induces π in $\Sigma(\mathbb{F}_{q^{18}})$ and so we cannot have that the linear set induced by $\Pi(\mathbb{F}_{q^9})$ intersects \mathcal{V}_1 in only 9 lines of $\Sigma(\mathbb{F}_{q^{18}})$. Let t be 3. If $\mu = 3$, then $d = 1$ and W is a 3-dimensional subspace. Reasoning as before, we get that $\Pi(\mathbb{F}_{q^3})$ induces a plane π of $\Sigma(\mathbb{F}_{q^3})$ intersecting \mathcal{V}_1 in $\Sigma(\mathbb{F}_{q^3})$ in three lines not through the

same point, hence Π induces a subplane $\text{PG}(2, q^2)$ disjoint from \mathcal{V}_1 in Σ , as shown in Proposition 3. Let now $\mu = 1$. Hence $d = 3$. Suppose that W is reducible, then there must be a component of W of dimension 3 and degree 1, getting again the previous case.

Suppose now that $d = 3 = t$, $\mu = 1$ and W is irreducible.

If W is a variety of $\text{PG}(5, q^3)$, then it is a variety of minimal degree (see [5, Corollary 18.12]) and the only possibility is that W is the Segre variety $\Sigma_{1,2}$, product of a line and a plane. The variety $\Sigma_{1,2}$ contains two ruling of maximal subspaces, one consisting of lines and the other of planes. A plane of $\Pi(\mathbb{F}_{q^3})$ induces a \mathbb{F}_{q^3} -linear set of rank 3 of $\Sigma(\mathbb{F}_{q^6})$. Such a linear set is either a subplane $\text{PG}(2, q^3)$ or the points of a line $\text{PG}(1, q^6)$, with only one of them with weight 2. Suppose that one of them, say π , induces a subplane. If a subplane $\text{PG}(2, q^3)$ is contained in \mathcal{V}_1 in $\Sigma(\mathbb{F}_{q^6})$, then all the $\text{PG}(2, q^6)$ containing it is contained in \mathcal{V}_1 and, as q is even, such a plane contains at least a point of π_N by Proposition 2. As π_N is setwise fixed by the automorphism of $\Sigma(\mathbb{F}_{q^6})$ induced by $\text{Gal}(\mathbb{F}_{q^6}/\mathbb{F}_{q^3})$, we have at least an intersection point in the subplane induced by π . Hence there exists $P \in \pi$ such that P induces a point of π_N . The subspace $\langle p, P^\sigma, P^{\sigma^2} \rangle \subset \Pi(\mathbb{F}_{q^3})$ is setwise fixed by $\text{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$, hence the subspace $\langle P, P^\sigma, P^{\sigma^2} \rangle \cap \Pi$ has the same dimension over \mathbb{F}_q and it has points of $\pi_N \subset \mathcal{V}_1$. Suppose that all the planes of $\Sigma_{1,2}$ induce lines of $\Pi(\mathbb{F}_{q^3})$, hence every plane of $\Sigma_{1,2}$ has a line that induces a point of weight 2 in $\Sigma(\mathbb{F}_{q^6})$. The same is true for the planes of $\Sigma_{1,2}^\sigma$ and $\Sigma_{1,2}^{\sigma^2}$. If these lines span $\Pi(\mathbb{F}_{q^3})$, we have that $\Pi(\mathbb{F}_{q^3})$ is a plane of $\Sigma(\mathbb{F}_{q^6})$ and hence Π is a plane of Σ . Suppose that all these lines are contained in the same three-dimensional space, hence there exists a line L of $\Sigma(\mathbb{F}_{q^6})$ with $3(q^3 + 1)$ points of \mathcal{V}_1 , hence $L \subset \mathcal{V}_1$. But L is setwise fixed by $\text{Gal}(\mathbb{F}_{q^6}/\mathbb{F}_{q^2})$, hence there exists a subline of L induced by Π contained in \mathcal{V}_1 . The last possibility is that W is a variety of $\text{PG}(4, q^3)$, hence a hypersurface of $\text{PG}(4, q^3)$ of degree 3. In this case, either one of $V(f)$ and $V(g)$ contains a hyperplane or there is a hyperplane H of $\text{PG}(5, q^3)$ such that $f \equiv g$ on H . In both cases, there exist $\alpha, \beta \in \mathbb{F}_q$ not both equal to zero such that $\alpha f + \beta g = h^{1+\sigma+\sigma^2}$, where h is the linear polynomial defining H . Let $f = x_1 y_1 z_1 + x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1 + a x_2 y_2 z_2 + f^*$ and $g = x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1 + a(x_1 y_2 z_2 + x_2 y_1 z_2 + x_2 y_2 z_1) + (a^2 + 1)x_2 y_2 z_2 + g^*$, so f^* and g^* contain monomials with at least one variable raised to the second power. Any linear combination of f and g cannot contain the monomials $x_1 x_2 y_1, x_1 x_2 y_2, x_1 x_2 z_1, x_1 x_2 z_2$. Let $h = a_1 x_1 + b_1 y_1 + c_1 z_1 + a_2 x_2 + b_2 y_2 + c_2 z_2$, then the coefficient of $x_1 x_2 y_1$ is $\text{Tr}(a_1 a_2^q b_1^{q^2} + a_1 a_2^{q^2} b_1^q)$, where $\text{Tr} : \mathbb{F}_{q^3} \rightarrow \mathbb{F}_q$ is the usual trace function. We

have that, in even characteristic, $Tr(a_1 a_2^q b_1^{q^2} + a_1 a_2^{q^2} b_1^q) = \begin{vmatrix} a_1 & a_2 & b_1 \\ a_1^q & a_2^q & b_1^q \\ a_1^{q^2} & a_2^{q^2} & b_1^{q^2} \end{vmatrix}$

and, by [12, Lemma 3.51], this matrix is singular if and only if $\{a_1, a_2, b_1\}$ is dependent over \mathbb{F}_q . Analogously, by the lack of $x_1 x_2 y_2, x_1 x_2 z_1, x_1 x_2 z_2$, we get that also $\{a_1, a_2, b_2\}$, $\{a_1, a_2, c_1\}$ and $\{a_1, a_2, c_2\}$ are dependent over \mathbb{F}_q . So either $b_1, b_2, c_1, c_2 \in \langle a_1, a_2 \rangle_{\mathbb{F}_q}$ or $\{a_1, a_2\}$ is dependent over \mathbb{F}_q . In the first case, we would have $\dim_{\mathbb{F}_q} \langle a_1, a_2, b_1, b_2, c_1, c_2 \rangle = 2$, but this is not possible. In fact, any linear combination $\alpha f + \beta g$ with $(\alpha, \beta) \neq (0, 0)$ contains at least one monomial of type $x_i y_j z_h$, as $f + f^*$ contains monomials not contained in $g + g^*$ and vice versa, and this implies that $\dim_{\mathbb{F}_q} \langle a_i, b_j, c_h \rangle = 3$, a contradiction to $\dim_{\mathbb{F}_q} \langle a_1, a_2, b_1, b_2, c_1, c_2 \rangle = 2$. So $\{a_1, a_2\}$ is dependent over \mathbb{F}_q . Analogously, the lack of $y_1 y_2 x_1, y_1 y_2 x_2, y_1 y_2 z_1, y_1 y_2 z_2$ implies that $\{b_1, b_2\}$ is dependent over \mathbb{F}_q and the lack of $z_1 z_2 x_1, z_1 z_2 x_2, z_1 z_2 y_1, z_1 z_2 y_2$ implies that $\{c_1, c_2\}$ is dependent over \mathbb{F}_q . So $a_2 = \lambda a_1$, $b_2 = \mu b_1$ and $c_2 = \nu c_1$, with $\lambda, \mu, \nu \in \mathbb{F}_q$, and $T := Tr(a_1 b_1^q c_1^{q^2} + a_1 b_1^{q^2} c_1^q) \neq 0$. The coefficient of $x_1 y_1 z_1$ in $\alpha f + \beta g$ is α , while it is T in $h^{1+\sigma+\sigma^2}$, hence $\alpha = T$. The coefficient of $x_1 y_1 z_2, x_1 y_2 z_1$ and $x_2 y_1 z_1$ in $\alpha f + \beta g$ is β , while they are, respectively, $\lambda T, \mu T, \nu T$ in $h^{1+\sigma+\sigma^2}$, hence $\lambda = \mu = \nu$ and $\lambda T = \beta$. Finally, the coefficient of $x_1 y_2 z_2$ is $\alpha + a\beta$ in $\alpha f + \beta g$ and it is $\mu\nu T = \lambda^2 T$ in $h^{1+\sigma+\sigma^2}$, so we get $1 + a\lambda + \lambda^2 = 0$, but $t^2 + at + 1$ is irreducible over \mathbb{F}_q by hypothesis. \square

Theorem 5. *Let \hat{G} be the group of collineations fixing \mathcal{V} (and hence \mathcal{V}_1). The planes disjoint from \mathcal{V}_1 form a unique orbit under the action of \hat{G} , and hence such a orbit consists of the planes inducing the Desarguesian spread.*

By Proposition 3, a plane π is disjoint from \mathcal{V}_1 if and only if $\pi \cap \mathcal{V}_1$ consists of three nonconcurrent lines through three points $P, P^\sigma, P^{\sigma^2}$ of \mathcal{V} over \mathbb{F}_{q^3} , with σ the \mathbb{F}_q -linear collineation induced by $Gal(\mathbb{F}_{q^6}/\mathbb{F}_{q^2})$. We have that $P^{\sigma^i} = v_2(R^{\sigma^i})$, $R \in PG(2, q^6)$, $i = 0, 1, 2$, where by abuse of notation we have denoted by σ also the \mathbb{F}_{q^2} -linear collineation of $PG(2, q^6)$ induced by $Gal(\mathbb{F}_{q^6}/\mathbb{F}_{q^2})$. If the points $R, R^\sigma, R^{\sigma^2}$ were collinear, then π would contain the image of the line through them, i.e. π would contain a conic of \mathcal{V} and hence $\pi \subset \mathcal{V}_1$. Hence $R, R^\sigma, R^{\sigma^2}$ are not collinear. Let $R = (x, y, z) \in PG(2, q^6)$, then $R, R^\sigma, R^{\sigma^2}$ are not collinear if and only if $\begin{pmatrix} x & y & z \\ x^{q^2} & y^{q^2} & z^{q^2} \\ x^{q^4} & y^{q^4} & z^{q^4} \end{pmatrix}$ is nonsingular and by [12, Lemma 3.51] this is equivalent to having $\{x, y, z\}$ independent over \mathbb{F}_{q^2} . Let $R' = (x', y', z') \in PG(2, q^6)$ be another point such that $R', R'^\sigma, R'^{\sigma^2}$ are not collinear, hence $\{x, y, z\}$ and $\{x', y', z'\}$ are two bases of \mathbb{F}_{q^6} considered as \mathbb{F}_{q^2} -vector space, so there exists

an element $g \in G = PGL(3, q^2)$ such that $\{x, y, z\}^g = \{x', y', z'\}$. As g and σ commute, $\forall g \in G$, we have that G is transitive on the sets $\{R, R^\sigma, R^{\sigma^2}\}$ with $R \in PG(2, q^6)$ such that $\{R, R^\sigma, R^{\sigma^2}\}$ is not contained in a line. This implies that the lifting of G , say \hat{G} , fixing \mathcal{V} acts transitively on the planes π of $PG(5, q^2)$ disjoint from \mathcal{V}_1 . This unique orbit hence contains the plane inducing the Desarguesian spread.

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